Bondi mass loss formula for axial symmetric systems in $f(R) = R + \lambda R^2$ gravity

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Motivation

The Bondi-Sachs formalism describes the flow of energy leaving an isolated gravitational system. An important prediction is the existence of a *gravitational wave memory effect.*

Moreover, at conformal infinity, it introduces the asymptotic symmetries known as the BMS group. This topic has been of great interest in the search for a quantum theory of gravity [2].

We intend to extend this framework to a more general family of scalar-tensor theories of gravity.
We start with Starobinsky gravity, a special case of $f(R)$ gravity. As we show, other cases can be covered by different choices of potentials in the Einstein frame.
Where possible, we attempt to automate the calculations using CAS, such that results can be generated different choices of the potential.

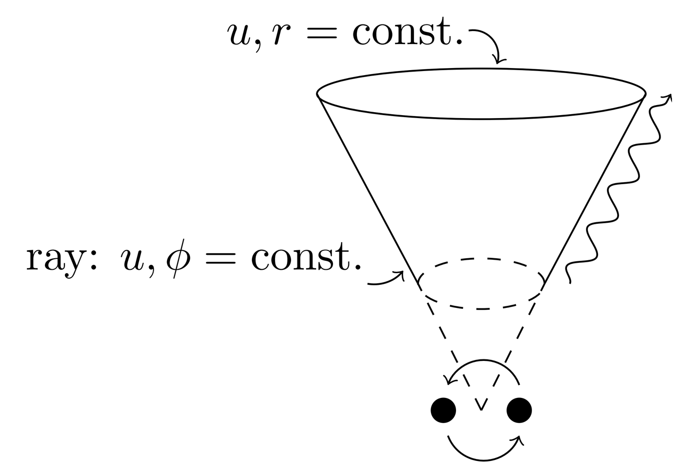

Illustration of the coordinate system with θ excluded.

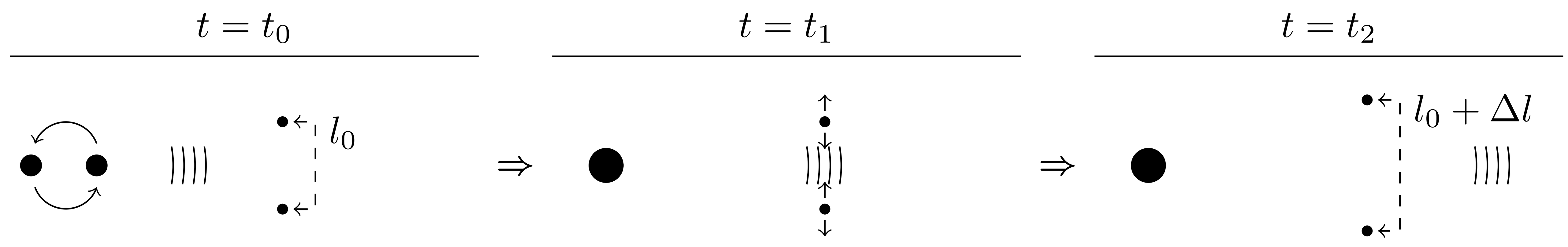

Illustration of the memory effect as burst of gravitational waves goes through space between free falling particles.

Bondi-Sachs formalism in GR

The formalism is based on the Bondi-Sachs metric

$$g = (A^2 r^2 e^{2\gamma} - B r^{-1} e^{2\beta})\,du^2 - 2e^{2\beta}\,du\,dr - 2Ar^2 e^{2\gamma}\,du\,d\theta + r^2(e^{2\gamma}\,d\theta^2 + e^{-2\gamma}\sin^2\theta\,d\phi^2)$$

It contains four free functions, is axially symmetric and the coordinates are chosen such that only the r coordinate varies along the light cones.
For the Ricci tensor, $R_{03} = R_{13} = R_{23} = 0$ hold identically. Also, by imposing $R_{ab} = 0$, we have

$$0 = r^{-2} e^{-2\beta} (r^2 R_{00})_r = 2r^{-1} e^{-2\beta} R_{01} = r^{-2} e^{-2\beta} (r^2 R_{02})_r$$

Thus, R_{00} and R_{02} have to be of the form $\alpha(u,\theta) r^{-2}$ and $R_{01} = 0$ identically. Finally, γ, β, A and B can be solved for using a negative power series ansatz iterating over $u = \text{const.}$ hypersurfaces like follows

1. $0 = R_{11} \Rightarrow \beta_r$
2. $0 = 2r^2 R_{12} \Rightarrow A_r$
3. $0 = r^2 R^3_3 e^{2\beta} - R_{22} e^{2(\beta-\gamma)} \Rightarrow B_r$
4. $0 = R^3_3 e^{2\beta} r^2 \Rightarrow \gamma_{ur}$

From R_{00} and R_{02} we have the Bondi mass and impulse loss formulae

$$M_u = -c_u^2 + \frac{1}{2}(c_{\theta\theta} + 3c_\theta \cot\theta - 2c)_u$$
$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u \cot\theta + c_u c_\theta$$

Hence, the development of the system is fully determined by the free parameters c, M and N. The function c_u describes the flow of energy out from the system and is called the "news function".

The main result of the original 1962 paper [1] thus states:

The mass of a system is constant if and only if there is no news. If there is news, the mass decreases monotonically as long as the news continues.

Path to Bondi-Sachs formalism in Starobinsky gravity

Starobinsky gravity is an alternative theory of gravity introduced originally to explain inflation [5]. It is described by the action

$$S_{\mathrm{S}} = \frac{1}{2\kappa^2}\int f(R)\sqrt{-g}\,d^4x, \quad f = R + \lambda R^2$$

This theory can be brought into a scalar-tensor form by a Legendre transformation $f(R) \mapsto V(\Psi)$

$$S_{\mathrm{J}} = \frac{1}{2\kappa^2}\int (R\Psi - V)\sqrt{-g}\,d^4x, \quad V = \frac{(\Psi - 1)^2}{4\lambda}$$

The resulting action S_{J} is expressed in the so called *Jordan frame*. The respective equations of motion are given by [3]

$$R_{ab} = \frac{1}{\Psi}\nabla_a \Psi_b + \frac{\kappa^2}{3\Psi} g_{ab} V + \frac{\kappa^2}{3} g_{ab} \frac{dV}{d\Psi}, \quad \Box\Psi = \frac{2\kappa^2}{3}\left(\Psi\frac{dV}{d\Psi} - 2V\right)$$

To determine the asymptotic fall-off conditions for the free functions, we transform the action into the *Einstein frame* and expand analogous to the existing work on the Brans-Dicke theory [6].

We introduce a new metric $\tilde{g}_{ab} = \Psi g_{ab}$, which yields the action

$$S_{\mathrm{E}} = \frac{1}{2\kappa^2}\int \left(\tilde{R} - \frac{1}{2}\tilde{g}^{ab}\varphi_a\varphi_b - \tilde{V}(\varphi)\right)\sqrt{-\tilde{g}}\,d^4x, \quad \tilde{V} = e^{-2\varphi/\sqrt{3}} V(e^{\varphi/\sqrt{3}})$$

and respective equations of motion

$$\tilde{R}_{ab} = \frac{1}{2}\varphi_a\varphi_b + \frac{1}{2}\tilde{V}\tilde{g}_{ab}, \quad \tilde{\Box}\varphi = \frac{d\tilde{V}}{d\varphi}$$

As seen, if one takes $\tilde{V} = 0$ the equations are equivalent to those of the Brans-Dicke theory in Einstein frame.

Future work and additional information

With the preparation thus far, the next step of our current plan is to reproduce the calculations of the Bondi-Sachs formalism for Starobinsky gravity and derive the Bondi mass loss formula. Then, we wish to drop axial symmetry and rewrite our results in more general tensor notation, while also considering more general potentials.

New revisions of this poster and the source code of the maxima script used to check the original paper of Bondi et al[1] can be found at:
`https://thomaslabs.org/research/erlangen2026.html`
When done, the source code for more general calculations will also be made publicly available.

References

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